Birth & Death of Stars

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Outline

- Star Formation
  - ➤ Self Gravity
  - ➤ Turbulence
  - ➤ Magnetic Field
- Death of Stars
  - ➢ White Dwarf
  - ➢ Neutron Star
  - ➢ Black Holes

General Understanding

- Fluid description
- Virial Theorem
- Jeans Instability
- Time scales
- Critical Mass

- Observations of nearby galaxies have shown, over a broad range of galactic environments and metallicities, that star formation occurs only in the molecular phase of the interstellar medium (ISM).
- Star formation is inextricably linked to the molecular clouds
- Interstellar gas converts from atomic to molecular only in regions that are well shielded from interstellar ultraviolet (UV) photons. UV photons are the dominant source of interstellar heating - only in these shielded regions does the gas become cold enough to be subject to Jeans instability
- ISM: roughly uniform gas, $n \sim 5 \times 10^6 m^{-3}$. Stars form in molecular clouds. Dimensions 10pc, density $\sim 5 \times 10^9 m^{-3}$, temperature $\sim\ 10K$.

Fluid Dynamics Equations - volume V bounded by surface S

- Continuity Equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

- Equation of motion - Euler's equation

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\nabla P + f_{ext}$$

- In presence of gravitation $F_{ext} = -\rho \nabla \phi$ where $\nabla^2 \phi = 4\pi G \rho$

Virial Theorem

- Consider the eqn. of motion at steady state *i.e.* $\frac{\partial}{\partial t} = 0$
- take $\vec{r}\times$.eqn. of motion and integrate over volume
- $\Rightarrow 2T + 3\Pi + W +$ surface integrals =0 where $T = \int_V \frac{1}{2}\rho v^2 dV$,
  $\Pi = \int_V P \cdot dV = \int_V nkTdV = \frac{2}{3}U, \quad W = -\int_V \rho\vec{r}\cdot\vec{\nabla}\phi$
- $\Rightarrow 2T + 2U + W = 2E + W = 0$ for finite systems; $W \propto -GM^2/R$
- For a slowly contracting star $T \sim 0 \Rightarrow W = -2U$
- $\Rightarrow U = W/2$ increases with contraction and star heats up
- rest half of W radiated away

Jeans Instability

Unperturbed system at steady state ρ_0, P_0, ϕ_0 *i.e.* $\frac{\partial}{\partial t} = 0$ and $\vec{v_0} = 0$

small perturbation $\Rightarrow \rho = \rho_0 + \rho_1$; $P = P_0 + P_1$; $\phi = \phi_0 + \phi_1$ and $\vec{v} = \vec{v_1}$

Neglecting higher order terms one gets for continuity eqn. and eqn. of motion

$\frac{\partial \rho_1}{\partial t} + \rho_0(\vec{\nabla} \cdot \vec{v_1}) = 0$; $\rho_0 \frac{\partial \vec{v_1}}{\partial t} = -{c_s}^2 \vec{\nabla} \rho_1 - \vec{\nabla} \phi_1$; $\quad \nabla^2 \phi_1 = 4\pi G \rho_1$; $\quad {c_s}^2 = \frac{\partial p}{\partial \rho}$

Taking time derivative of first eqn. and then combining with all three

$$\frac{\partial^2 \rho_1}{\partial t^2} - {c_s}^2 \nabla^2 \rho_1 = -4\pi G \rho_0 \rho_1$$

Plane wave solution $\rho_1 = e^{i(\vec{k} \cdot \vec{x} - \omega t)}$

$\Rightarrow \omega^2 = {c_s}^2 k^2 - 4\pi G \rho_0$

for ${c_s}^2 k^2 < 4\pi G \rho_0 \Rightarrow \omega$ becomes imaginary

$\Rightarrow k < k_j = \frac{\sqrt{4\pi G \rho_0}}{c_s}$; $\lambda_j = \frac{2\pi}{k} = c_s \sqrt{\frac{\pi}{G \rho_0}}$ $\Rightarrow$ Jeans Length

$M_j = \frac{4\pi}{3} \rho_0 (\lambda_j/2)^2 = \frac{\pi^{(5/2)}}{6} \frac{{c_s}^3}{G^{3/2} {\rho_0}^{1/2}}$ $\Rightarrow$ Jeans Mass

One can also write $\frac{R}{c_s} \geq (G\rho_0)^{-1/2}$ (check dimension)

R.H.S $\rightarrow$ Free fall time under gravity for negligible pressure

L.H.S $\rightarrow$ time taken for a sound wave to cross the system $\Rightarrow$ Jeans criterion implies that sound can not traverse the system (pressure can not operate) in time to prevent the collapse

Jeans Instability

Taking particle mass $m = 2 \times 10^{-24}$gm, $c_s = 260$m/s at
$T = 10k$ $\lambda_j = 1.0pc(\frac{T}{10k})^{1/2}(\frac{n}{10^3 cm^{-3}})$
$M_j = 26.0M \odot (\frac{T}{10k})^{3/2}(\frac{n}{10^3 cm^{-3}})$

- Dense molecular clouds containing more than few tens of solar mass of gas is unstable
- As unstable region collapses $\rightarrow$ density rises
- If cloud is optically thin to cooling radiation then temperature remains roughly constant
- As collapse proceeds density increases, Jeans mass decreases
- Collapsing cloud fragments into lower and lower mass pieces, each collapsing in its on free fall time
- Collapse ends when fragments become very dense and optically thick

More on protostars

- Release of grav energy $\rightarrow$ increase temperature and excites molecules and atoms
- Energy released can be absorbed by dissociation of H_2, $\epsilon_D =$ 4.5 eV and ionisation of H atom $\epsilon_I =$ 13.6 eV
- amount of energy released $\frac{M}{2m_H}\epsilon_D + \frac{M}{m_H}\epsilon_I$
- Approximate radius of protostar from $GM(\frac{1}{R_2} - \frac{1}{R_1}) \approx \frac{1}{m_H}(\frac{\epsilon}{2} + \epsilon_I)$
- As an example consider a protostar of $1M_\odot$ with initial radius 10^{15} m with Jeans density $\sim 10^{-16}$ Kg m^{-3}
- Dissociation + Ionization $\sim 3 \times 10^{39}$J
- Then radius after gravitational contraction $R_2 \sim 10^{11}$m
- time scale for this contraction $t_{dyn} \sim 20000$ Years

Approach to Hydrostatic equilibrium

- After all H is ionised, internal pressure rises, contraction slows down - system approaches hydrostatic equilibrium
- Thermal K.E. of protons and electrons $E_k \sim \frac{3}{2}KT\frac{M}{\mu m_H} = \frac{3KTM}{m_H}$, $\mu =$ mean molecular weight =0.5 for the present case
- At the end of collapse $E_{grav} \sim -\frac{GM^2}{R_2} \sim -\frac{M}{m_H}(\frac{\epsilon}{2} + \epsilon_I)$
- From Virial theorem $E_k + E_{grav} = 0$
- Protostar approaches equilibrium $KT \sim \frac{\epsilon_D + 2\epsilon_I}{12} \sim 2.6eV$, Temperature $T \sim 30000^0$K - independent of mass of protostar
- Gravitational energy radiated away on a Thermal (Kelvin - Helmholtz) time scale
- Thermal time scale - time required by the star to radiate away all its energy in absence of nuclear source $\tau_{KH} = \frac{E_{int}}{L} \approx \frac{E_{grav}}{L} \approx \frac{GM^2}{2RL}$
- L = rate of energy loss $-\frac{dE}{dt}$
- $\tau_{KH} \approx 1.5 \times 10^7 (\frac{M}{M_\odot})^2 \frac{R_\odot}{R} \frac{L_\odot}{L}$ Yr.

How far contraction proceeds?

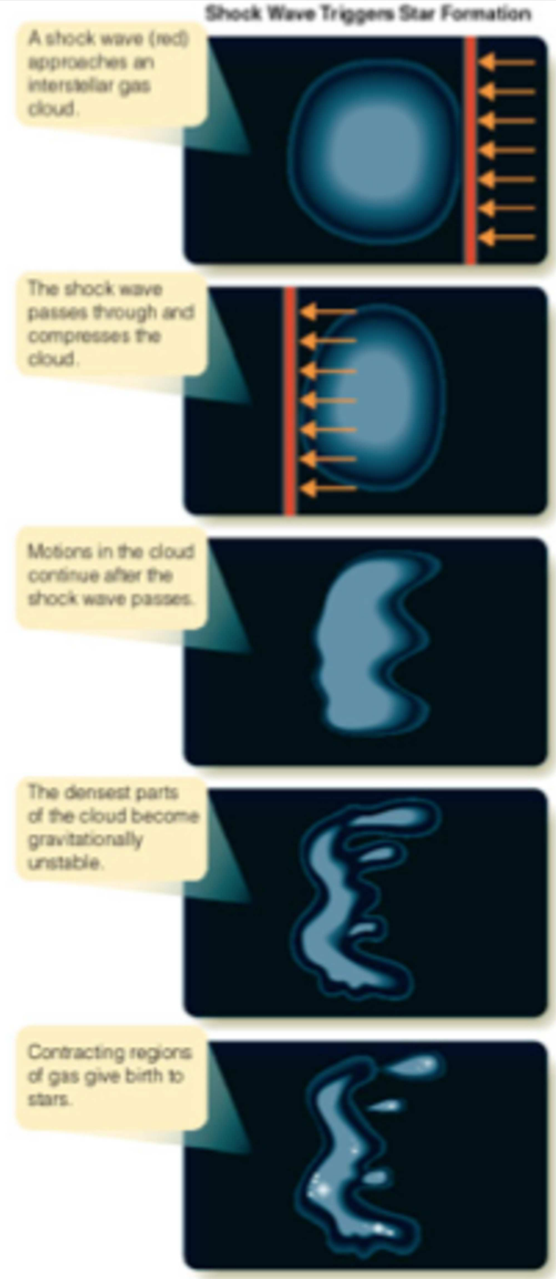
- Classical Mechanics breaks down when wave function of neighbouring electrons overlap
- $r = \lambda = \frac{h}{m_e v}$ - de Broglie
- $\frac{1}{2} m_e v^2 = \frac{3}{2} KT \Rightarrow r = \frac{h}{(3 m_e KT)^{1/2}}$
- $\Rightarrow \rho = \frac{\mu m_H}{4\pi r^3/3} = \mu m_H \frac{(m_e KT)^{3/2}}{h^3}$
- At this point further increase in density does not affect the temperature
- Virial Theorem $KT \approx \frac{GM\mu m_H}{3R} \approx G\mu m_h M^{2/3} \rho^{1/3}$
- Substituting ρ $KT_{max} \approx (\frac{G\mu m_H{}^{8/3} m_e}{h^2}) M^{4/3}$
- $T_{max} \approx 10^7 (\frac{M}{M_\odot})^{4/3}$
- For $M < 0.08 M_\odot$ T_{max} is too small to start nuclear reaction
- Quantum tunnelling may allow the barrier penetration around de-Broglie separation
- here K.E. = Coulomb potential and the corresponding separation will be $\lambda = \frac{h^2}{2\mu m_H Z_i Z_j e^2}$
- So one may write the earlier condition in terms of $Z_i Z_j$ of the participating nuclei

Nuclear time scale

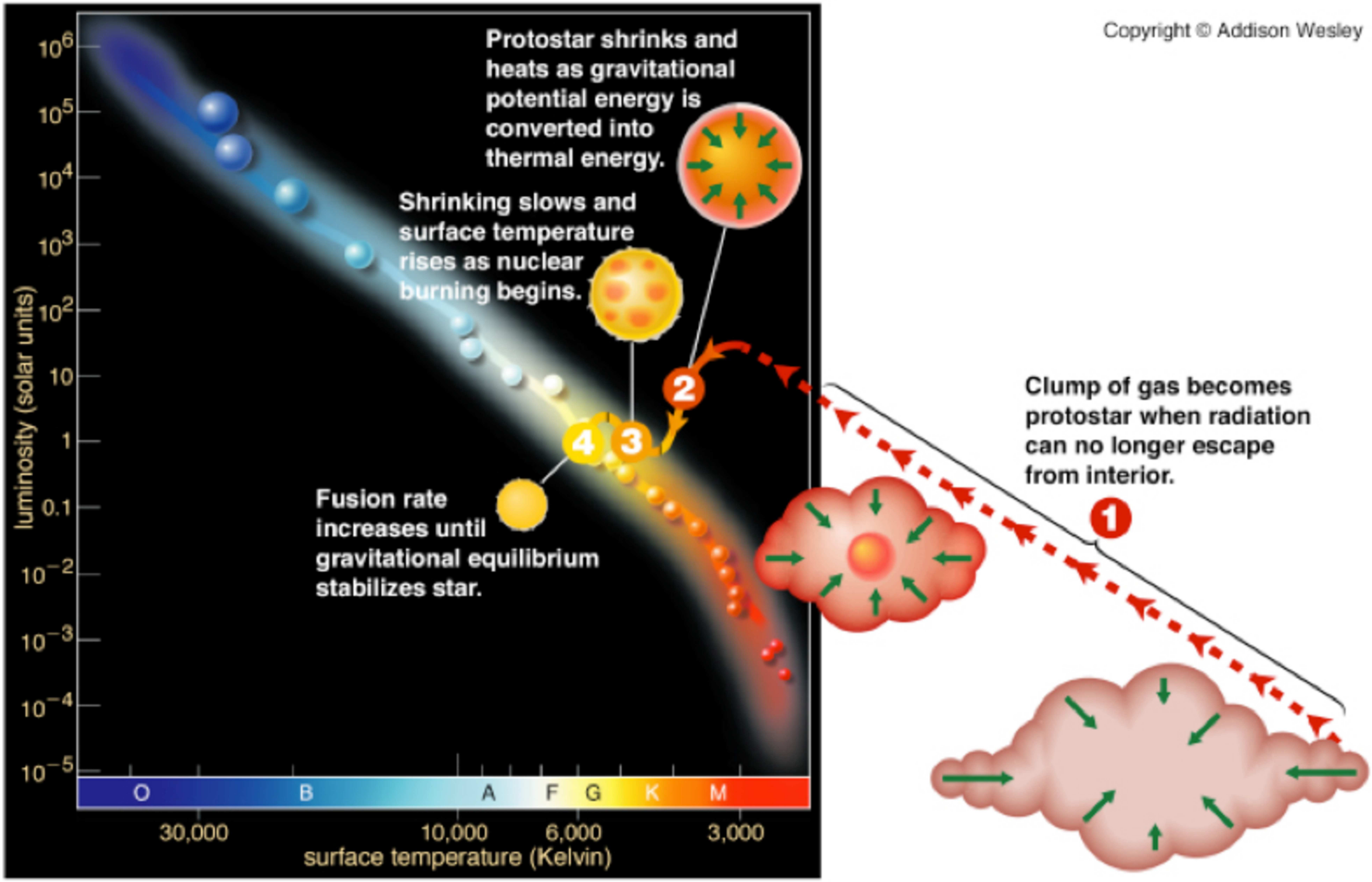
Star can remain in thermal equilibrium for as long as its nuclear fuel supply lasts. Associated time scale is Nuclear time scale. Since Nuclear fuel (say H) is burned into ash (say He), it is also the time scale on which composition changes in the stellar interior occur

- ϕ = fraction of rest mass of reacting nuclei that converts into energy
- f_n= fraction of the mass of the star which may serve as nuclear fuel
- Total nuclear energy supply $E_n = \phi M_n c^2 = \phi f_n M c^2$
- In thermal equilibrium $L = L_n = \frac{dE_n}{dt}$
- $\tau_n = \frac{E_n}{L}$
- For Sun like star, 70% hydrogen and only 10% of its participates
- $\Rightarrow \tau_n \approx \frac{0.007 M_\odot c^2}{L_\odot} \approx 10^{11}$ Years
- $\tau_n >> \tau_{KH} >> \tau_{dyn}$
- Rate of nuclear reaction determines the pace of stellar evolution and stars may be assumed to be in hydrostatic and thermal equilibrium most of their lives

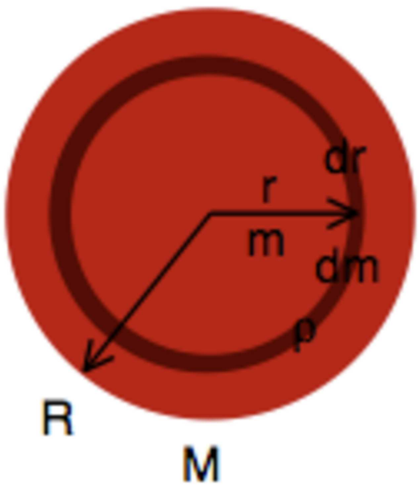
Supernova shocks trigger SF



Then a Protostar is born



Description of a star in spherical symmetry



Let r be the distance from the center
Density as function of radius is $\rho(r)$

If m is the mass *interior* to r, then:

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr$$

Differential form of this equation is: $dm = 4\pi r^2 \rho dr$

Two **equivalent** ways of describing the star:

- Properties as f(r): e.g. temperature T(r)
- Properties as f(m): e.g. T(m)

Second way often more convenient, because (ignoring mass loss) total mass M of the star is fixed, while radius R evolves with time.

Stellar Structure

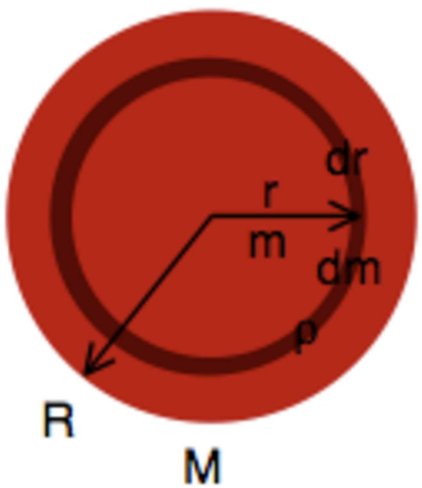
For an isolated, static, spherically symmetric star, four basic laws / equations needed to describe structure:

- **Conservation of mass**
- **Conservation of energy** (at each radius, the change in the energy flux equals the local rate of energy release)
- **Equation of hydrostatic equilibrium** (at each radius, forces due to pressure differences balance gravity)
- **Equation of energy transport** (relation between the energy flux and the local gradient of temperature)

Basic equations are supplemented by:

- **Equation of state** (pressure of a gas as a function of its density and temperature)
- **Opacity** (how transparent it is to radiation)
- **Nuclear energy generation rate** as $f(\rho, T)$

Conservation of Mass



Let r be the distance from the center
Density as function of radius is ρ(r)

Let m be the mass *interior* to r, then
conservation of mass implies that:

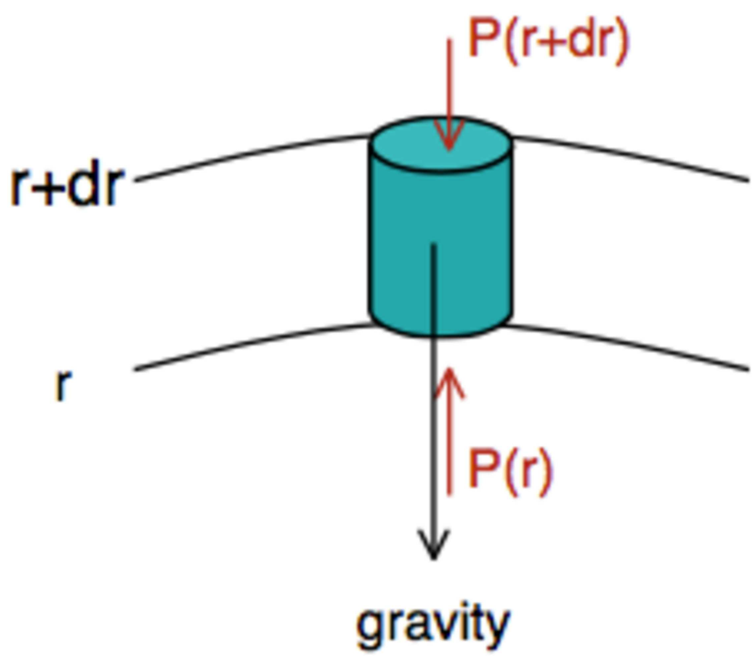
$$dm = 4\pi r^2 \rho dr$$

Write this as a differential equation:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

1st stellar structure equation

Equation of Hydrostatic Equilibrium



Consider small cylindrical element between radius r and radius r + dr in the star.

Surface area = dS
Mass = Δm

Mass of gas in the star at Smaller radii = m = m(r)

Radial forces acting on the element:

Gravity (inward): $$F_g = -\frac{Gm\Delta m}{r^2}$$

Pressure (net force due to difference in pressure between upper and lower faces):

$$F_p = P(r)dS - P(r+dr)dS$$

$$= P(r)dS - \left[P(r) + \frac{dP}{dr} \times dr\right]dS$$

$$= -\frac{dP}{dr}drdS$$

Mass of element: $\Delta m = \rho drdS$

Applying Newton's second law (`F=ma') to the cylinder:

$$\Delta m\ddot{r} = F_g + F_p = -\frac{Gm\Delta m}{r^2} - \frac{dP}{dr}drdS$$

acceleration = 0 everywhere if star static

Setting acceleration to zero, and substituting for Δm:

$$0 = -\frac{Gm\rho drdS}{r^2} - \frac{dP}{dr}drdS$$

<u>**Equation of hydrostatic equilibrium**</u>:

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

2nd stellar structure equation

If we use enclosed mass as the dependent variable, can combine these two equations into one:

$$\frac{dP}{dm} = \frac{dP}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

← alternate form of hydrostatic equilibrium equation

Equation of State in Stars

Interior of a star contains a mixture of ions, electrons, and radiation (photons). For most stars (exception very low mass stars and stellar remnants) the ions and electrons can be treated as an ideal gas and quantum effects can be neglected.

Total pressure:

$$P = P_I + P_e + P_r$$
$$= P_{gas} + P_r$$

- P_I is the pressure of the ions
- P_e is the electron pressure
- P_r is the radiation pressure

Gas Pressure

The equation of state for an ideal gas is: $P_{gas} = nkT$

n is the number of particles per unit volume; $n = n_I + n_e$, where n_I and n_e are the number densities of ions and electrons

In terms of the mass density ρ: $P_{gas} = \dfrac{\rho}{\mu m_H} \times kT$

…where m_H is the mass of hydrogen and μ is the average mass of particles in units of m_H. Define the **ideal gas constant**:

$$R \equiv \frac{k}{m_H} \quad \Rightarrow \quad P_{gas} = \frac{R}{\mu} \rho T$$

Determining Mean Molecular Weight μ

μ will depend upon the composition of the gas and the state of ionization. For example:

- Neutral hydrogen: $\mu = 1$
- Fully ionized hydrogen: $\mu = 0.5$

In the central regions of stars, OK to assume that all the elements are fully ionized.

Denote abundances of different elements per unit mass by:

- **X** hydrogen - mass m_H, one electron
- **Y** helium - mass $4m_H$, two electrons
- **Z** the rest, `metals', average mass Am_H, approximately (A / 2) electrons per nucleus

If the density of the plasma is ρ, then add up number densities of hydrogen, helium, and metal nuclei, plus electrons from each species:

	H	**He**	**metals**
Number density of nuclei	$\frac{X\rho}{m_H}$	$\frac{Y\rho}{4m_H}$	$\frac{Z\rho}{Am_H}$
Number density of electrons	$\frac{X\rho}{m_H}$	$\frac{2Y\rho}{4m_H}$	$\approx \frac{A}{2} \times \frac{Z\rho}{Am_H}$

$$n = \frac{\rho}{m_H}\left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right] = \frac{\rho}{\mu m_H}$$

...assuming that A >> 1

$$\Rightarrow \quad \mu^{-1} = 2X + \frac{3}{4}Y + 2Z$$

Radiation Pressure

For blackbody radiation: $$P_r = \frac{1}{3} a T^4$$

…where a is the *radiation constant*:

$$a = \frac{8\pi^5 k^4}{15 c^3 h^3} = \frac{4\sigma}{c}$$

$$= 7.565 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$$

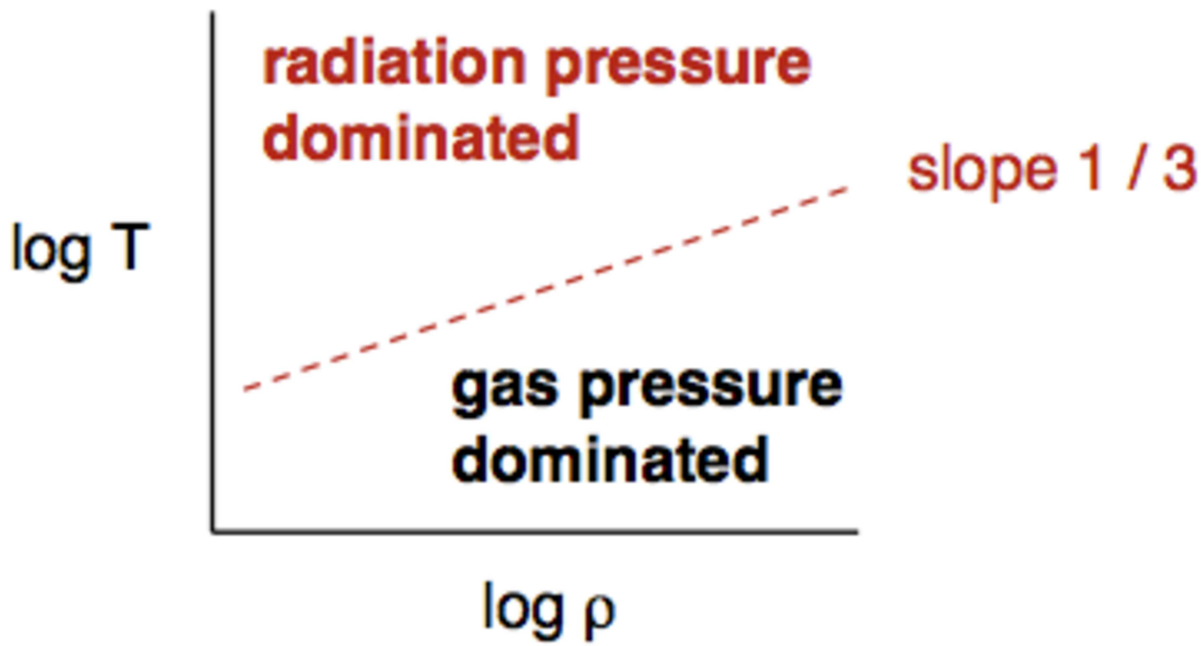
$$= 7.565 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}$$

In which stars are gas and radiation pressure important?

$$P_{gas} = \frac{R}{\mu}\rho T$$

$$P_r = \frac{1}{3}aT^4$$

equal when: $T^3 = \frac{3R}{a\mu}\rho$



Gas pressure is most important in **low mass stars**
Radiation pressure is most important in **high mass stars**

Conditions in the Solar core

A detailed model of the Sun gives core conditions of:

- $T = 1.6 \times 10^7$ K
- $\rho = 150$ g cm^{-3}
- $X = 0.34$, $Y = 0.64$, $Z = 0.02$ (note: hydrogen is almost half gone compared to initial or surface composition!)

$$\mu^{-1} = 2X + \frac{3}{4}Y + 2Z \Rightarrow \mu = 0.83$$

Ideal gas constant is $R = 8.3 \times 10^7$ erg g^{-1} K^{-1}

Ratio of radiation pressure to gas pressure is therefore:

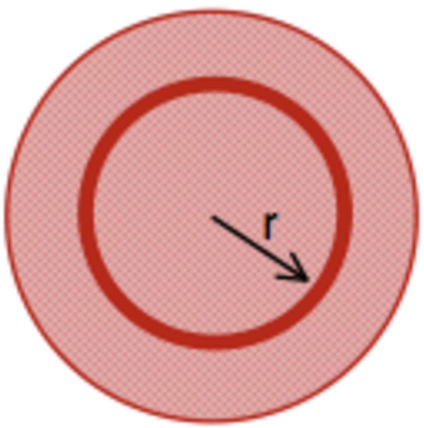
$$\frac{P_r}{P_{gas}} = \frac{\frac{1}{3}aT^4}{\frac{R}{\mu}\rho T} = \frac{\mu a}{3R} \times \frac{T^3}{\rho} = 7 \times 10^{-4}$$

Radiation pressure is not at all important in the center of the Sun under these conditions

Equation of energy generation

Assume that the star is in thermal equilibrium - i.e. at each radius the gas is neither heating up nor cooling down with time.

Let the **rate of energy generation per unit mass** be q (with units erg s^{-1} g^{-1}). Then:



Shell, mass $dm = 4\pi r^2 \rho dr$
Luminosity at r: L(r)
Luminosity at r+dr: L(r)+dL

$$dL = 4\pi r^2 \rho dr \times q$$

$$\frac{dL}{dr} = 4\pi r^2 \rho q$$

4^{th} stellar structure equation

Summary: equations of stellar structure

At radius r in a static, spherically symmetric star:

- density ρ
- enclosed mass m (mass at *smaller radii)*
- temperature T
- luminosity L

$$\frac{dm}{dr} = 4\pi r^2 \rho$$ **Mass conservation**

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$ **Hydrostatic equilibrium**

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$ **Energy transport due to radiation (only)**

$$\frac{dL}{dr} = 4\pi r^2 \rho q$$ **Energy generation**

4 equations in 4 unknowns - enough for a solution once we know P(ρ,T), κ and q.

What About Rotation?

Gravitational potential at distance r from a point mass m is:

$$\phi = -\frac{Gm}{r}$$

`Average' element of gas in a star is about distance R From the center, and has mass M interior to its radius, where R and M are the stellar radius and total mass. Typical potential is thus:

$$\phi \sim -\frac{GM}{R} \Rightarrow \text{gravitational binding energy } E_{grav} \sim M\phi \sim -\frac{GM^2}{R}$$

Solar rotation period is about P = 27 days. Angular velocity:

$$\Omega = \frac{2\pi}{P} \approx 2.7 \times 10^{-6}\ \mathrm{s}^{-1}$$

Rotation energy is of the order of:

$$E_{rotation} \sim M\Omega^2 R^2$$

Compare magnitude of gravitational and rotational energy:

$$\beta = \frac{E_{rotation}}{|E_{grav}|} = \frac{M\Omega^2 R^2}{GM^2/R} = \frac{\Omega^2 R^3}{GM} \sim 2 \times 10^{-5}$$

Depends upon **square** of rotation velocity

…even rotation rates much faster than the Sun ought to be negligibly small influence on structure.

What About Magnetic Fields?

Magnetic fields in sunspots are fairly strong, of the order of kG strength. Suppose same field fills Sun:

$$E_{magnetic} = \text{Volume} \times \text{Energy density}$$

$$= \frac{4}{3}\pi R^3 \times \frac{B^2}{8\pi} = \frac{B^2 R^3}{6}$$

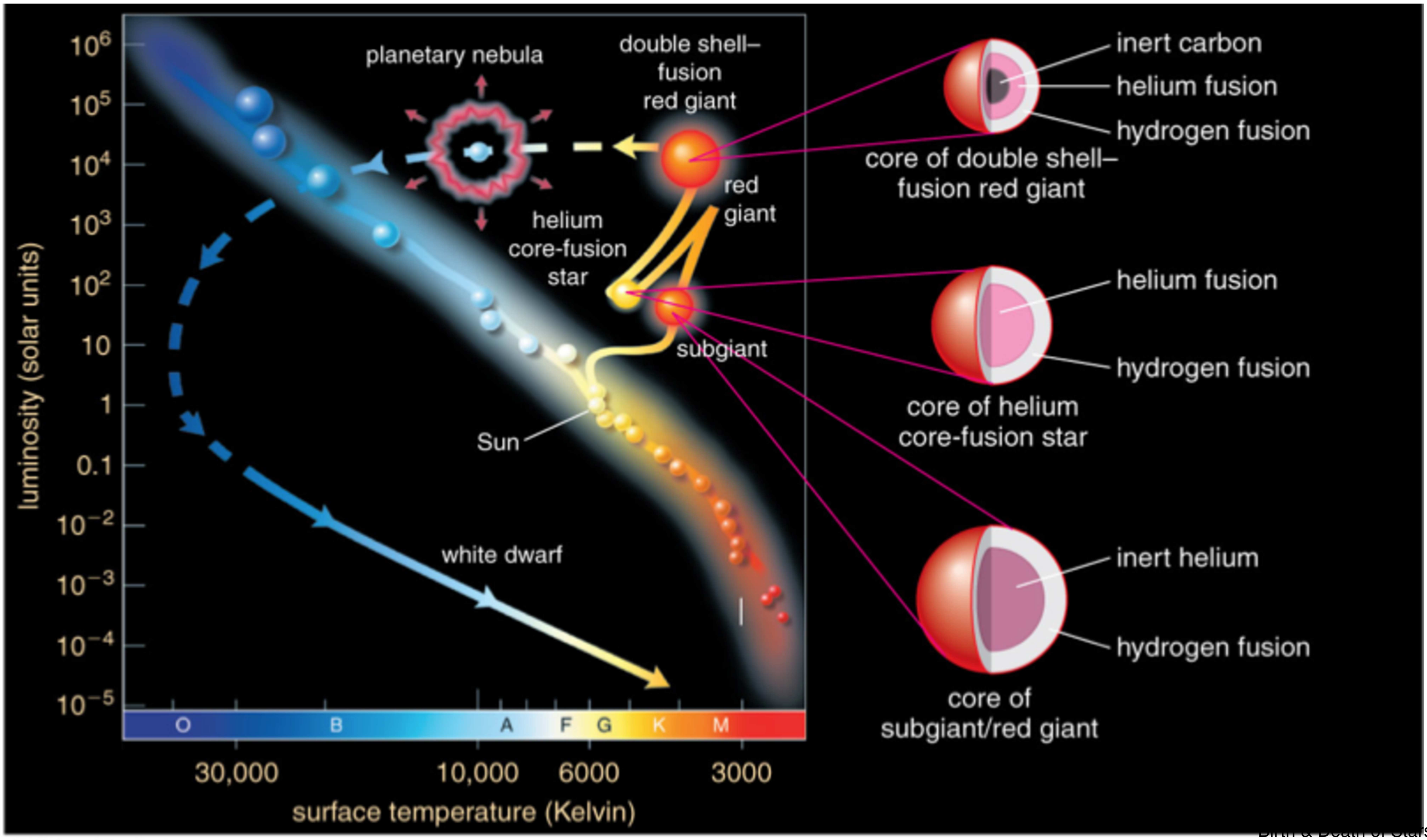
Ratio to gravitational energy is:

$$\frac{E_{magnetic}}{\left|E_{grav}\right|} = \frac{B^2 R^3 / 6}{GM^2 / R} = \frac{B^2 R^4}{6GM^2} \sim 10^{-11}$$

Estimates suggest that unless something really weird is going on (e.g. Sun rotates super-fast on the inside but not at the surface) magnetic fields / rotation are too small to seriously affect assumption of spherical symmetry.

H-R Diagram

- made independently by Enjar Hertzsprung and Henry Norris Russell
- graph of luminosity (or absolute magnitude) versus temperature (or spectral class)
- Most of the stars in Main sequence - H burning to He



Hydrogen Burning

- PP-I cycle, $T > 4 \times 10^6$ K: $4\ {}^1H \Rightarrow {}^4He$:

  $2\ ({}^1H + {}^1H = {}^2D + \beta^+ + \nu_e + 0.42\ \text{MeV})$

  $(\beta^+ + \beta^- = \gamma + 1.02\ \text{MeV})$

  $2\ ({}^1H + {}^2D = {}^3He + \gamma + 5.49\ \text{MeV})$

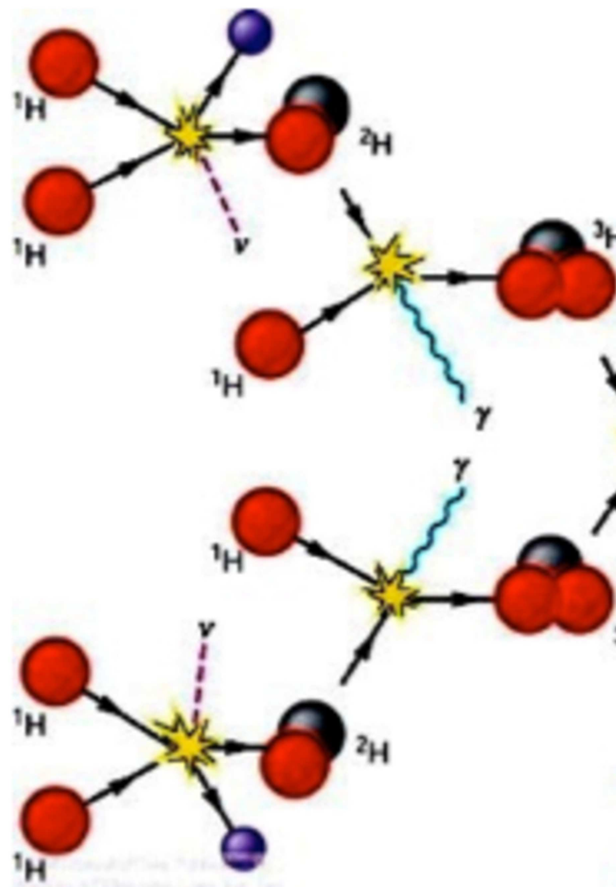
  ${}^3He + {}^3He = {}^4He + 2\ {}^1H + 12.86\ \text{MeV}$

- PP-II cycle, $T > 14 \times 10^6$ K:

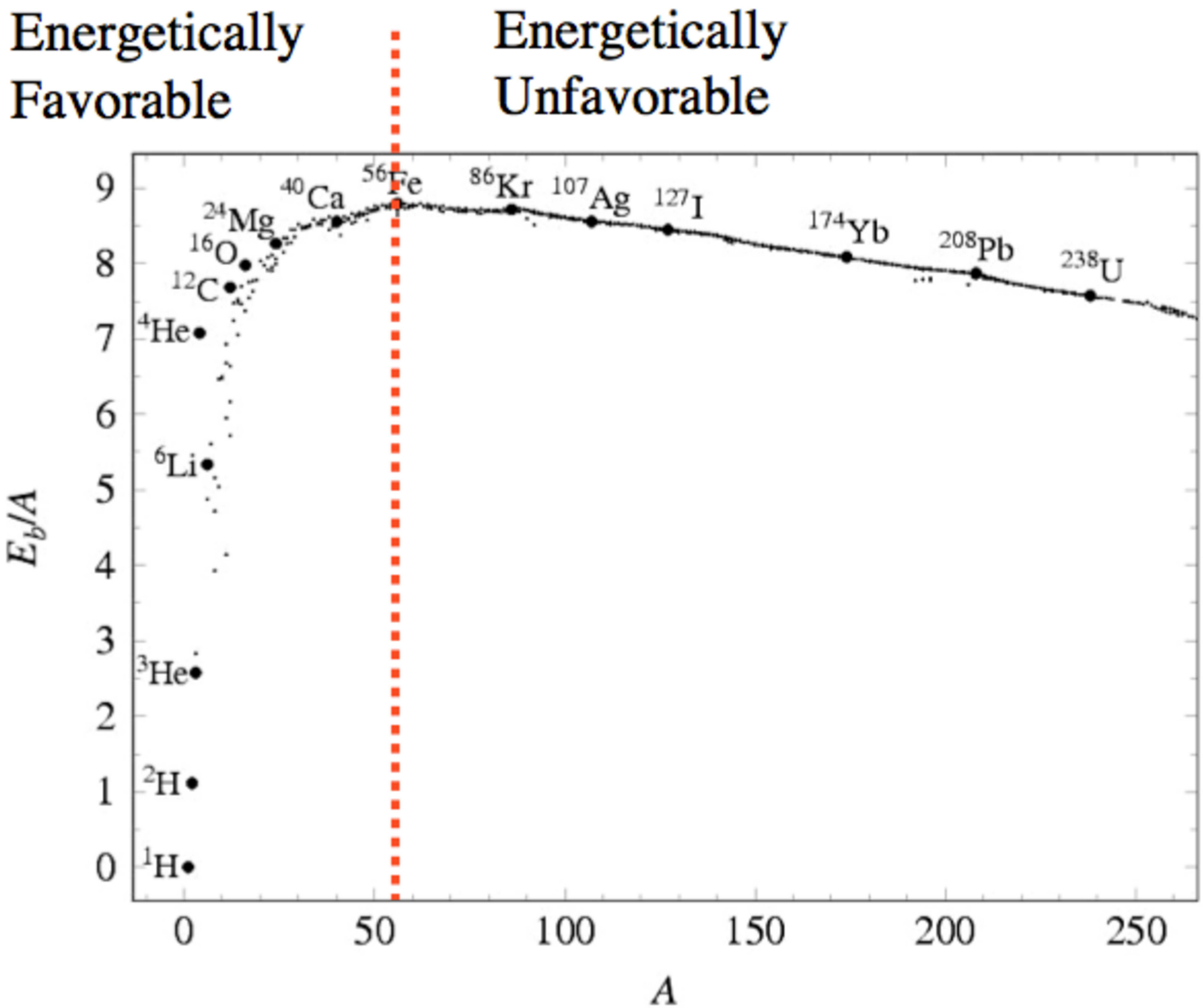
  ${}^3He + {}^4He = {}^7Be + \gamma$

  ${}^7Be + \beta^- = {}^7Li + \nu_e$

  ${}^7Li + {}^1H = 2\ {}^4He$



Binding Energy Per Nucleon vs. Atomic Number



Further evolution

- hydrogen atoms in the core of the star that fuse together to create helium, start to run out and fusion begins to slow down , system becomes out of balance
- When hydrogen fusion ceases in the core, the star collapses inward; this causes the layer just outside the core to become so hot and dense that hydrogen fusion begins in this outer layer.
- Energy produced by hydrogen fusion in this layer just outside the core causes the rest of the star to expand into a giant star.
- While the exterior layers expand, the helium core continues to contract and eventually becomes hot enough (100 million Kelvin) for helium to begin to fuse into carbon and oxygen
- stars with mass $< 8M_\odot$ and $> 8M_\odot$ have different fates

Stellar Fusion Requirements

Fusion	Fusion By-product	Minimum Core Temperature	Minimum Core Density	Minimum Stellar Mass*
Hydrogen	He	13 million K	100 gm/cc	0.08 solar masses
Helium	C, O	100 million K	100,000 gm/cc	0.5 solar masses
Carbon	O, Ne, Mg, Na	500 million K	200,000 gm/cc	4 solar masses
Neon	O, Mg	1.2 billion K	4 million gm/cc	about 8 solar masses
Oxygen	Mg, Si, S, P	1.5 billion K	10 million gm/cc	about 8 solar masses
Silicon	Si, S, Ar, Ca, Ti, Cr, Fe, Ni	around 3 billion K	30 million gm/cc	about 8 solar masses

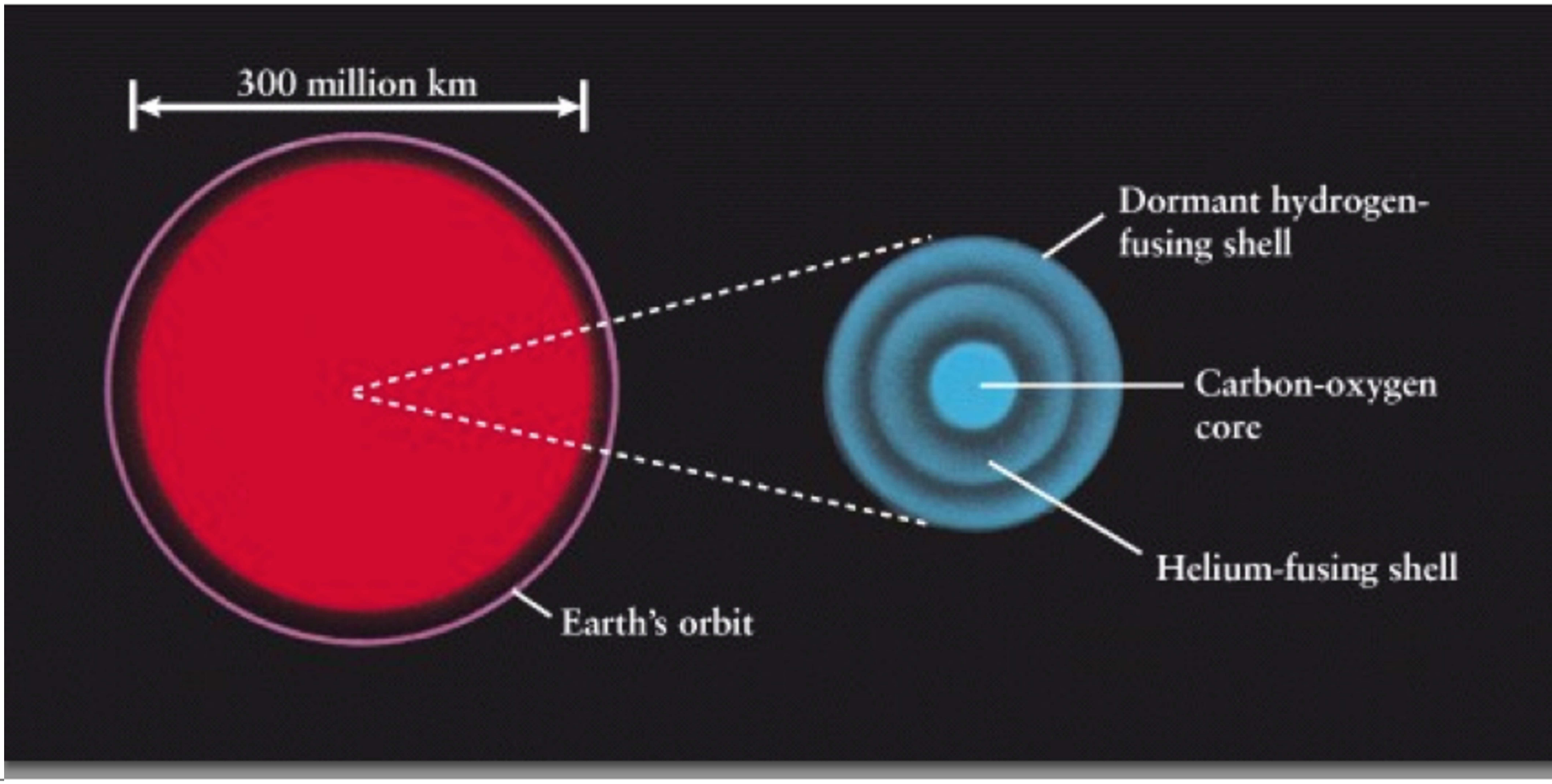
*Stars with masses less than this will not able to reach the temperature and density criteria for the fusion to begin. For the neon, oxygen and silicon fusion, the mass estimates are very approximate and uncertain.

Stellar Nucleosynthesis

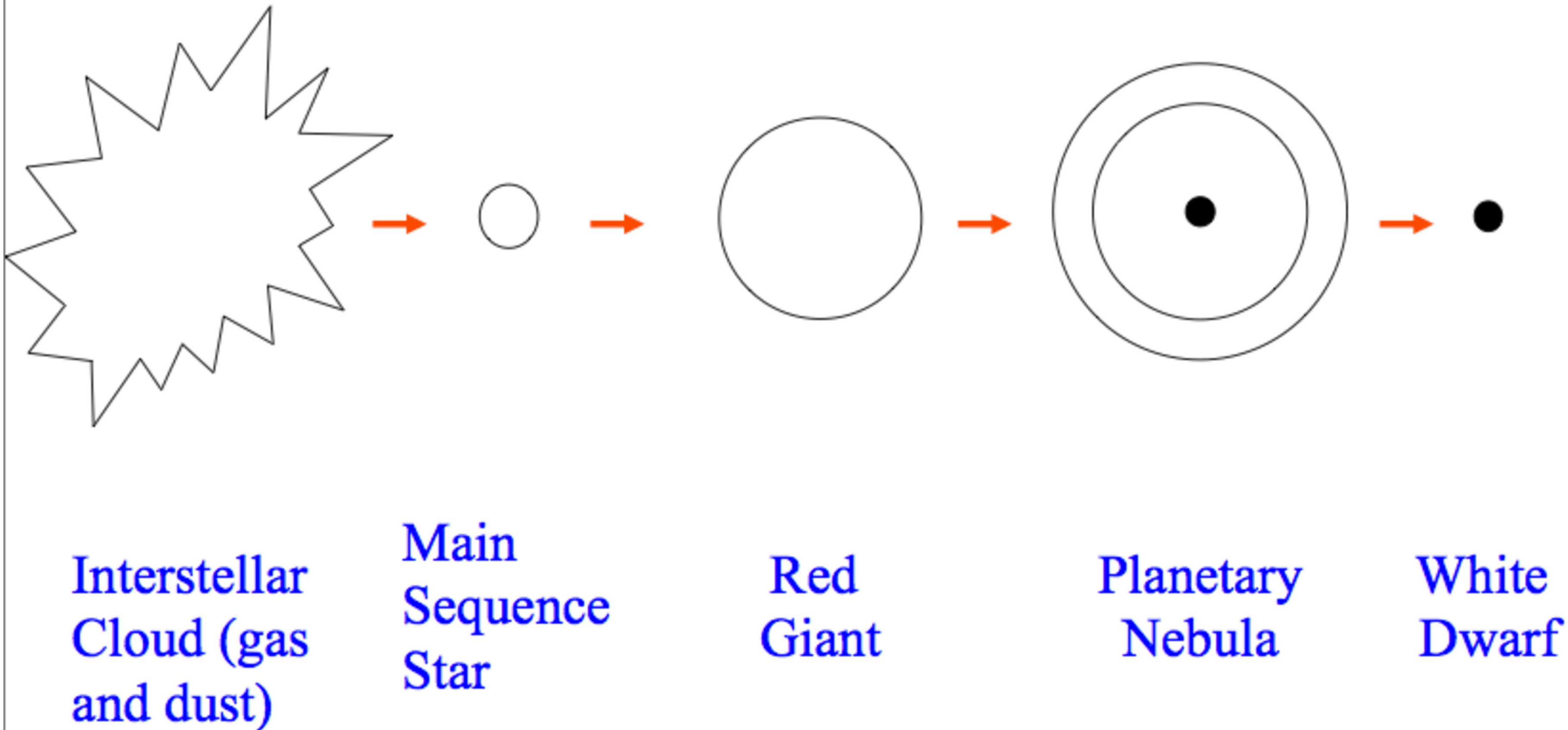
Evolutionary Time Scales for a 15 M_{sun} Star

Fused	Products	Time	Temperature
H	^{4}He	10^{7} yrs.	4×10^{6} K
^{4}He	^{12}C	Few X 10^{6} yrs	1×10^{8} K
^{12}C	^{16}O, ^{20}Ne, ^{24}Mg, ^{4}He	1000 yrs.	6×10^{8} K
^{20}Ne +	^{16}O, ^{24}Mg	Few yrs.	1×10^{9} K
^{16}O	^{28}Si, ^{32}S	One year	2×10^{9} K
^{28}Si +	^{56}Fe	Days	3×10^{9} K
^{56}Fe	Neutrons	< 1 second	3×10^{9} K

Example of a low-mass giant: its outer layers and core



Fate of low mass stars



Compact Objects - Degenerate Stars

- Non-degenerate stars - Pressure driven by K.E. of gas particles $\frac{P}{\rho c^2} = f(T)$
- Degenerate stars - Pressure a consequence of Pauli principle $\frac{P}{\rho c^2} = f(\rho)$
- For an ideal gas $T \rightarrow 0$, $P \rightarrow 0$, no equilibrium possible
- For an ideal gas in equilibrium distribution function $f = \frac{1}{exp[(E-\mu)/kT] \pm 1}$; + for fermions and - for bosons
- No. density n, Energy density ϵ and Pressure are $n = \frac{N}{V} = \frac{g}{h^3} 4\pi \int \frac{p^2 dp}{exp[(E-\mu)/kT] \pm 1}$
$\epsilon = \frac{E_{tot}}{V} = \frac{g}{h^3} 4\pi \int E \frac{p^2 dp}{exp[(E-\mu)/kT] \pm 1}$ $P = \frac{1}{3} \frac{g}{h^3} 4\pi \int pv(p) \frac{p^2 dp}{exp[(E-\mu)/kT] \pm 1}$
- **Non degenerate non-relativistic gas**: $E = E_{NR} = p^2/2m$; For very large temperatures $f \rightarrow$ Maxwell - Boltzman distribution; $n \approx \frac{g}{h^3} 4\pi e^{\eta} \int_0^{\infty} \frac{p^2 dp}{exp[p^2/2mkT]}$; $\eta = \mu/kT$
- taking $x = p^2/2mkT$ and $mkTdx = dp$, $n = \frac{g}{h^3}(2\pi mkT)^{3/2} e^{\eta}$
- use same procedure, $\epsilon = \frac{3}{2} nkT$; $P = nkT = \frac{2}{3}$
- **non degenerate relativistic case**: p/mc » 1 and $E \approx pc$, $n = 8\pi g(\frac{kT}{hc})^3 e^{\eta}$,
$\epsilon = 24g(\frac{kT}{hc})^3 kTe^{\eta} = 3nkT$
- $\Rightarrow P = nkT = \frac{1}{3}\epsilon$

- **Fully degenerate case**: $T \to 0$; $\mu/kT \to \infty$; $\eta \to \infty$; $\mu \to E_F$ - the fermi energy
- Distribution function becomes $f(E) = 1 \quad E \leq E_F = \mu$ and $f(E) = 0 \quad E > E_F = \mu$
- Consider $F_n(\eta) = \int_0^\infty \frac{x^n\, dx}{exp[x-\eta]+1}$, $lim_{\eta\to\infty} F_n(\eta) = \int_0^\eta x^n dx = \frac{1}{n+1}\eta^{n+1}$
- **Nonrelativistic**: $n = \frac{g}{h^3} 2\pi(2mkT)^{3/2} \frac{2}{3}\eta^{3/2}$
- $\epsilon = \frac{g}{h^3} 2\pi(2mkT)^{3/2} kT \frac{2}{5}\eta^{5/2}$; Substituting n from above $\epsilon = \frac{3h^2}{10m}(\frac{3}{4\pi g})^{3/2}(n)^{5/3}$
- $\Rightarrow P \sim n^{5/3} \sim (Y_F\rho)^{5/3}$; Y_F is concentration of fermions per baryons
- **Fully relativistic**: $n = 4\pi g(\frac{kT}{hc})^3 \frac{1}{3}\eta^3$
- $\epsilon = 4\pi g(\frac{kT}{hc})^3 kT \frac{1}{4}\eta^4 \Rightarrow \epsilon = \frac{3}{4}(\frac{3}{4\pi g})^{1/3} hc(n)^{4/3}$
- $\Rightarrow P \sim n^{4/3} \sim Y_F\rho^{4/3}$

What is white dwarf?

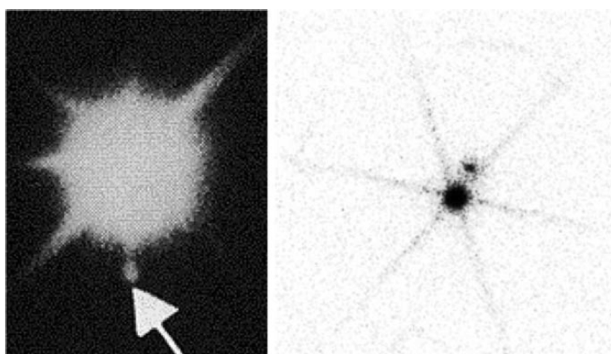
- White Dwarfs are the first example in theory of stellar structure of macroscopic consequences of quantum and relativistic effects taking place at microscopic level.
- White Dwarfs are characterized by too low luminosities and too small radii for their mass. The White Dwarf Sirius B has: $L = 1/400 L_\odot$, $M = 1 M_\odot$, $R = 1/40 R_\odot$ and $T_{eff} =$ 10000 K.
- Density of white dwarfs is very high going from $10^5 to 10^8$ g cm^{-3}. Gas of electrons must be strongly degenerate.
- Material is fully ionized, and made of free electrons and nuclei. Total pressure and total energy density are given by the sum of the contributions by nuclei and electrons. Radiation can be neglected.
- The electron gas is assumed to fully degenerate everywhere in the star but for a thin layer close to the surface.
- At the typical density of 10^6 g cm^{-3} the kinetic energy of electrons is about 0.15 Mev, whereas that of nuclei at the same density and temperatures of about $10^6 - 10^7$ K (typical values in white dwarfs interiors) is only about 1 - 10 Kev, we can neglect the pressure due to nuclei
- Gravitation is now balanced by the electron degeneracy pressure

White Dwarfs

- A core with remaining mass less than 1.4 $M_\odot$.
- These tiny star remnants are approximately the size of planet Earth
- One cubic centimeter (like a sugar cube) of a White Dwarf star would weigh several tons.
- Maximum mass of the White Dwarf is around 1.4 $M_\odot$ as given by Chandrasekhar limit
- Polytropic Equation of State $P = K\rho^\Gamma$ where Γ=5/3 for non-relativistic particles and Γ=4/3 for relativistic particles
- Considering relativistic degenerate electron gas the equilibrium between gravitation and electron degeneracy pressure gives
  $R = 3.347 \times 10^4 \mathrm{Km}(\frac{\rho_c}{10^6 gcm^{-3}})^{-1/3}(\frac{Y_e}{0.5})^{2/3}$ & $M = M_{Ch} = 1.457 M_\odot (\frac{Y_e}{0.5})^2$
- mass is independent of both central density and radius
- Increase on central density reduces the radius - more compact
- Isolatetd White dwarfs cool and become black dwarf but White dwarfs in a binary have different fate

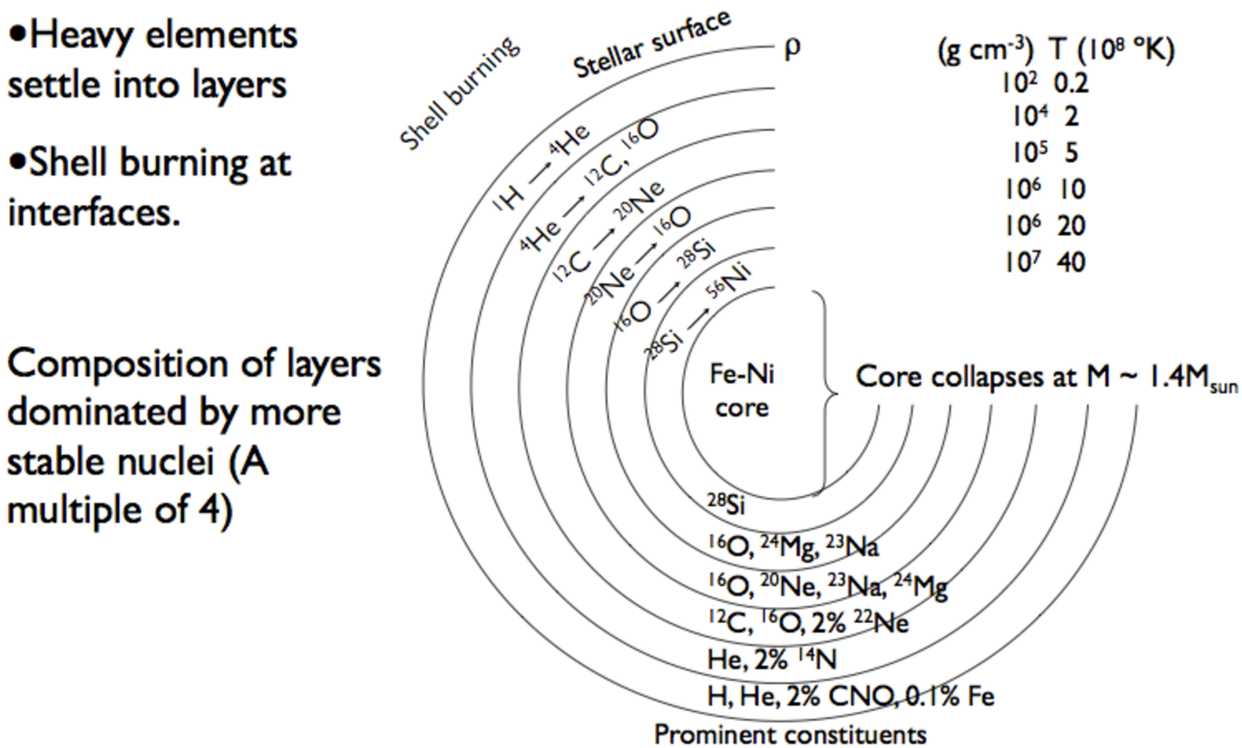
White Dwarf - Observation

- The first white dwarf $\rightarrow$ a companion star to Sirius, a bright star near the constellation Canis Major.
- 1844, astronomer Friedrich Bessel noticed a slight back and forth motion of Sirius, as if orbited by an unseen object.
- In 1863, mystery finally resolved by optician Alvan Clark and it was found to be a white dwarf. This pair is now referred to as Sirius A and B, B being the white dwarf.
- The black-body spectrum of Sirius B peaks at 110 nm, corresponding to a temperature of 27,000 K
- Optical spectra of white dwarfs $\rightarrow$ classified according to dominant element in the atmosphere. DA: strong hydrogen lines, DB: strong He I lines, DO: strong He II lines, DC: no strong lines (continuous) spectrum, DZ: strong metal lines (excluding carbon), DQ: strong carbon lines.
- If trace elements are seen, then they must be of recent origin (e.g. accretion from the ISM, comets, etc.).
- 65 catalogued isolated magnetic white dwarfs having magnetic field between 30KG to 1000MG.

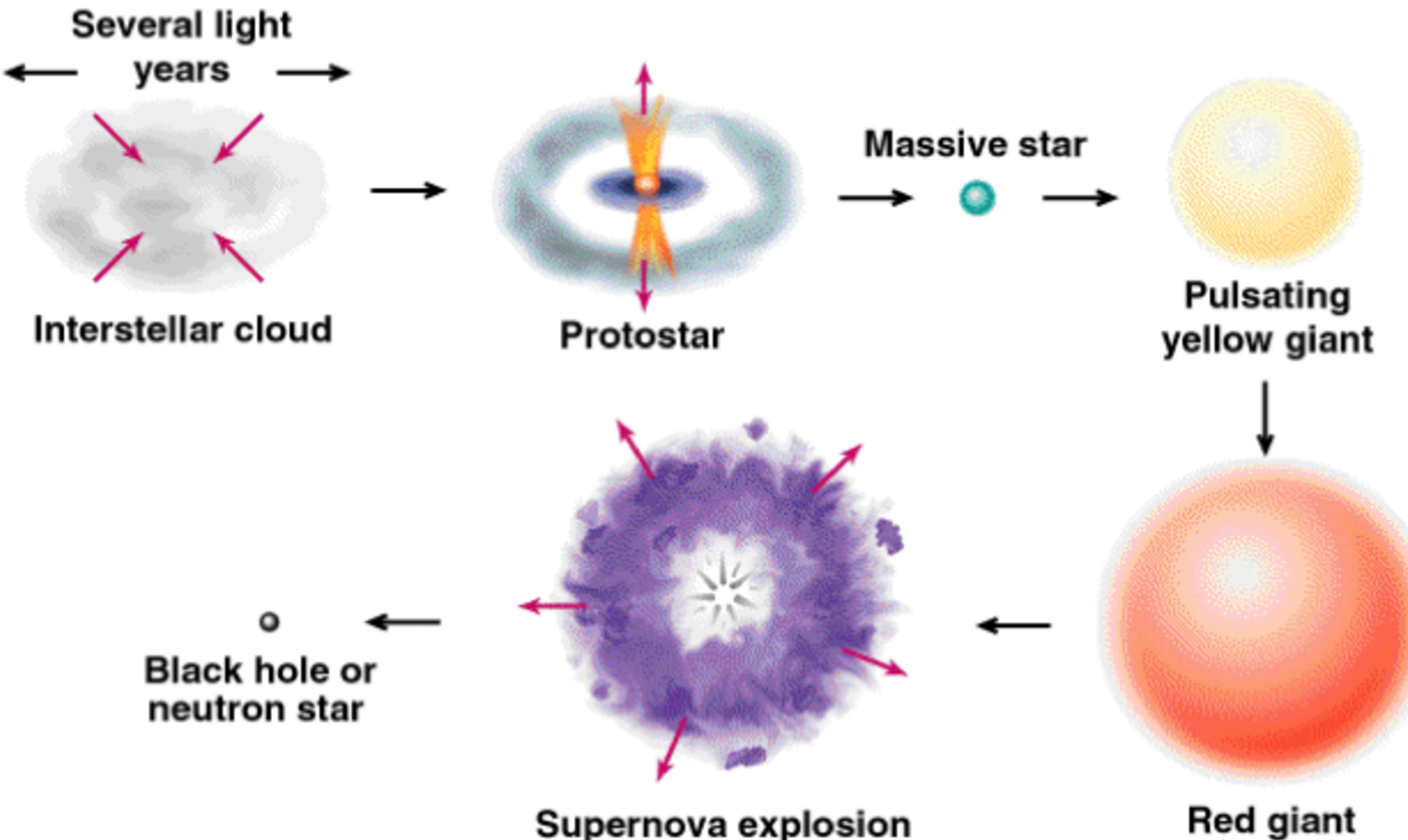


Fate of high mass stars

- Fusion in the core continues through many more stages than for low mass stars
  Heavier elements are produced; carbon, oxygen, neon, silicon, and so on up to iron
- Higher density $\Rightarrow e^- + p \rightarrow n + \nu$, more abundance of neutrons, gravity balanced by mainly Neutrons ; hence Neutron Stars



Life of a High Mass Star



Toward Collapse

- End of Si burning - ^{56}Fe core - T $\sim 10^9$ ^{0}K Central core of $\sim 1.5M_\odot$ - supported by electron degeneracy - $\gamma \sim 4/3$
- Two different effects that drive the core to collapse - (a) Photo-dissociation of Fe - uses binding energy of nuclei - lowers the pressure (b) Rise in density - electron capture - lower electron contribution to pressure
- **Photo dissociation**: $\gamma + {}^{56}{}_{26}Fe \leftrightarrow 13\alpha + 4n$, Energy required $Q^2 = c^2(13m_\alpha + 4m_n - m_{Fe}) = 124.4$ MeV, Chemical equilibrium $\rightarrow \mu_F e = 13\mu_\alpha + 4\mu_n$
- For typical existing conditions, nuclei and nucleons are non-degenerate $\frac{\mu_i - m_i c^2}{kT} = ln[\frac{n_i}{g_i}(\frac{h^2}{2\pi m_e kT})^{3/2}]$; Statistical weight $g_i = \sum_i (2I_r + 1)e^{-E_r/kT}$, I_r = spin of rth excited state, E_r = energy above the ground state
- For $T \leq 1$ MeV, $g_\alpha = 1$ (ground state, $I = 0$), $g_n = 1/2$ (free), $g_F e = 1.4$ (ground state + lowest excited state)
- Substituting the μ_i value in the Chemical equilibrium condition $\frac{{n_\alpha}^{13}{n_n}^4}{n_{Fe}} = \frac{{g_\alpha}^{13}{g_n}^4}{g_{Fe}}(\frac{kT}{2\pi\hbar^2})^{24}(\frac{{m_\alpha}^{13}{m_n}^4}{m_{Fe}})^{3/2}e^{-Q/kT}$
- $\Rightarrow$ **SAHA equation** for the equilibrium ratio of α and neutrons to Fe Nuclei
- Dissociation of 50% of Iron content needs $T_9 \sim 11$ - present in the core - collapse expected to set in

Neutronization

- **Thermal** $e^+ + e^- \to (W,Z) \to \nu + \bar{\nu}$;
- $e^- + \gamma \to (W,Z) \to e^- + \nu + \bar{\nu}$
- $e^- + (Z,A) \to (W,Z) \to (Z,A) + e^- + \nu + \bar{\nu}$
- **Netronization**
- $e^- + (Z,A) \to W \to \nu_e + (Z-1,A)$
- $e^- + p \to W \to \nu_e + n$
- Neutronization $\to$ reduce Y_e $\to$ decrease in electron pressure
- **Neutrino processes** - Neutrino scattering, and absorption
- ALL these processes along with all possible stable and unstable nuclei need to be considered for a successful collapse simulation
- Still an active field of research

Neutron Stars

- Prediction of the existence of neutron stars as a possible endpoint of stellar evolution was independent of observations.
- Following the discovery of the neutron by Chadwick, it was realized by many people that at very high densities electrons would react with protons to form neutrons via inverse beta decay.
- first proposed by Walter Baade and Fritz Zwicky to be the collapsed remains of a massive star after it has exploded as a supernova.
- Detected in 1967 by Cambridge University graduate student Jocelyn Bell. She found a radio source with a regular on-off-on cycle of exactly 1.3373011 seconds. Today, we know pulsars are rapidly spinning neutron stars.